International Journal of Modern Physics A, Vol. 0, No. 00 (1996) 0000–0000
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arXiv:hep-th/9512020v3 11 Nov 1998

HIGHER-ORDER DIFFERENTIAL OPERATORS ON A LIE GROUP AND QUANTIZATION

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ABSTRACT

This talk is devoted mainly to the concept of higher-order polarization on a group, which is introduced in the framework of a Group Approach to Quantization, as a powerful tool to guarantee the irreducibility of quantizations and/or representations of Lie groups in those anomalous cases where the Kostant-Kirilov co-adjoint method or the Borel-Weyl-Bott representation algorithm do not succeed.

1. Group Approach to Quantization

We start with a Lie group $\tilde{G}$ which is a $U(1)$-bundle with base manifold M. A one-form $\tilde{\theta} \equiv \Theta$ on $\tilde{G}$ will be naturally selected (see below) among the components of the left-invariant, Lie Algebra valued, Maurer-Cartan 1-form, in such a way that $\Theta(\tilde{X}_0) = 1$ and $L_{\tilde{X}_0}\Theta = 0$, where $\tilde{X}_0$ is the vertical vector field. Θ will play the role of a connection 1-form associated with the $U(1)$-bundle structure [1,2].

Let us consider firstly the case of a Lie group $\tilde{G}$ which is a central extension of a Lie group G by $U(1)$, defining a principal bundle $\pi : \tilde{G} \to G$. Such a central extension is characterized by a Lie group 2-cocycle $\xi : G \times G \to R$ in terms of which the group law $\tilde{g}'' = \tilde{g}' \tilde{*} \tilde{g}$ is written as $(g'', \zeta'') = (g' * g, \zeta'\zeta e^{i\xi(g',g)})$, or the

corresponding Lie algebra 2-cocycle $\Sigma : \mathcal{G} \times \mathcal{G} \to R$, where $\mathcal{G}$ is the Lie algebra of G. Thus, there is a unique, up to an exact 1-form $d\lambda$, left-invariant 1-form Θ on $\tilde{G}$ such that $d\Theta|_e = \tilde{\Sigma}$, where $\tilde{\Sigma}$ is the pull-back of Σ by π. The ambiguity in the definition of Θ only corresponds to the freedom of adding a coboundary $\delta\lambda(g', g) = \lambda(g' * g) - \lambda(g') - \lambda(g)$ to the cocycle. The fuction λ proves to be a function with no linear terms in any canonical co-ordinate system at the identity.

Now, we consider the case of a symmetry Lie group G with trivial cohomology, $H^2(G, U(1)) = 0$, but wearing a principal H-bundle structure (H=$U(1)$ or R) with projection $\pi : G \to M$, and let $\{\sigma_{\alpha\beta}\}$ be a Čech cocycle (which is unique save for a Čech coboundary) on a open covering $\{U_\alpha\}$ of M, defining this fibration. The composition $\{\sigma_{\alpha\beta} \circ \pi\}$ turns out to be a Lie group cocycle (a coboundary in fact) [3] defining a trivial central extension $\tilde{G} \equiv G \tilde{\otimes} U(1)$ whose Lie algebra structure constant are, nevertheless, non-trivially modified, thus defining a connection 1-form Θ as in the previous case. This type of not-so-trivial central extensions will be called pseudo-extensions and the corresponding coboundaries, pseudo-cocycles. These coboundaries constitute a subgroup of the group of coboundaries, the pseudo-cohomology group, in correspondence (one-to-one at the Lie algebra level) with the true cohomology group of the contraction (in the sense of Inönü and Wigner) of G with respect to the subgroup H [4,5].

In the general case, including infinite-dimensional semi-simple Lie groups, for which the Whitehead lemma does not apply, the principal bundle with connection $(\tilde{G}, \Theta)$ should be constructed by applying both procedures to G, i.e. the group law for $\tilde{G}$ will contain cocycles as well as pseudo-cocycles (see [6,3,7]).

We then start from a Lie group $\tilde{G}$ with connection 1-form Θ defined as before, verifying $i_{X_0}\Theta = 1$, X_0 being the infinitesimal generator of $U(1)$, or the fundamental vector field of the principal bundle. Since Θ is left-invariant it will be preserved ($L_{X^R}\Theta = 0$) by all right-invariant vector fields (generating finite left translations) on $\tilde{G}$. These vector fields are candidates to be infinitesimal generators of unitary transformations. However, to define the space of functions on which they should act, we first notice that by requiring

$$L_{X_0}\psi = i\psi \tag{1}$$

we select equivariant functions on $\tilde{G}$, which are associated with sections of the $U(1)$-bundle $\tilde{G}$ over G. To make the action of the right-invariant vector fields on equivariant functions irreducible, any operator commuting with them must be trivialized, and this is achieved by polarization conditions.

Two vector fields $\tilde{X}^L, \tilde{Y}^L$ (or the corresponding group parameters in local co-ordinates) are conjugated if $\tilde{\Sigma}(\tilde{X}^L, \tilde{Y}^L) \neq 0$. We will restrict ourselves to finite-dimensional Lie groups or infinite-dimensinal ones possessing a countable basis of generators for which, for arbitrary fixed $\tilde{X}^L$, $\tilde{\Sigma}(\tilde{X}^L, \tilde{Y}^L) = 0$ except for a finite number of vectors $\tilde{Y}^L$ (finitely non-zero cocycle). In these cases the skewsymmetric form $\tilde{\Sigma}$, seen as a pairing $\tilde{\Sigma} : \tilde{\mathcal{G}} \times \tilde{\mathcal{G}} \to R$, can always be taken to normal form defining canonically conjugate pairs of vector fields as well as those without symplectic

character. For instance, for the Galilei group the non-symplectic generators would be those associated with time, spatial rotations or the vertical generator itself. The intersection of $Ker\tilde{\Sigma} \equiv Kerd\Theta$ and $Ker\Theta$ (horizontality) in $\mathcal{X}^L(\tilde{G})$ defines a vector subspace, which proves to be a subalgebra, the **characteristic subalgebra** $\mathcal{G}_C$ generating the characteristic module of Θ.

A **first-order polarization** or just **polarization** $\mathcal{P}$ is defined as a maximal horizontal left subalgebra. The horizontality condition means only that the $U(1)$ generator is excluded from the polarization, a fact which will be relevant for further generalizations.

A polarization may have non-trivial intersection with the characteristic subalgebra. We say that a polarization is **full (or regular)** if it contains the whole characteristic subalgebra. We also say that a polarization is **symplectic** if it contains "half" of the symplectic vector fields, i.e. one of each pair of canonically conjugate vector fields.

Among all complex-valued functions on $\tilde{G}$ we select the **wave functions** as those which are constant along maximal isotropic integral surfaces of the polarization, i.e. $X^L\Psi = 0, \forall X^L \in \mathcal{P}$, and satisfy the equivariance condition $L_{X_0}\psi = i\psi$.

The Harmonic Oscillator

For a simple example, let us consider the case of the harmonic oscillator. The quantization group [1] is a symmetry group $\tilde{G}_{(m,\omega)}$ which goes to the centrally extended Galilei group as $\omega \to 0$. To simplify the posterior search for a polarization, we write the group law in (otherwise standard) coordinates $c \equiv \sqrt{\frac{m}{2\omega}}(\omega x + iv), c^* \equiv \sqrt{\frac{m}{2\omega}}(\omega x - iv)$, and $\eta \equiv e^{i\omega t}$. The group law is:

$$\begin{aligned} c'' &= c'\eta^{-1} + c \\ c^{*\prime\prime} &= c^{*\prime}\eta + c^* \\ \eta'' &= \eta'\eta \\ \zeta'' &= \zeta'\zeta e^{\frac{i}{2}[ic'c^*\eta^{-1} - ic^{*\prime}c\eta]} \end{aligned} \tag{2}$$

For the left- and right-invariant vector fields we find ($X_0 \equiv \Xi$):

$$\begin{array}{lcllcl} \tilde{X}^L_\eta &=& i\eta\frac{\partial}{\partial\eta} - ic\frac{\partial}{\partial c} + ic^*\frac{\partial}{\partial c^*} & \tilde{X}^R_\eta &=& i\eta\frac{\partial}{\partial\eta} \\ \tilde{X}^L_c &=& \frac{\partial}{\partial c} - \frac{i}{2}c^*\Xi & \tilde{X}^R_c &=& \eta^{-1}\frac{\partial}{\partial c} + \frac{i}{2}\eta^{-1}c^*\Xi \\ \tilde{X}^L_{c^*} &=& \frac{\partial}{\partial c^*} + \frac{i}{2}c\Xi & \tilde{X}^R_{c^*} &=& \eta\frac{\partial}{\partial c^*} - \frac{i}{2}\eta c\Xi \\ \tilde{X}^L_\zeta &=& i\zeta\frac{\partial}{\partial\zeta} \equiv \Xi & \tilde{X}^R_\zeta &=& i\zeta\frac{\partial}{\partial\zeta} \equiv \Xi \end{array} \tag{3}$$

with commutation relations

$$\begin{aligned} \left[\tilde{X}^L_c, \tilde{X}^L_{c^*}\right] &= i\Xi \\ \left[\tilde{X}^L_\eta, \tilde{X}^L_c\right] &= i\tilde{X}^L_c \\ \left[\tilde{X}^L_\eta, \tilde{X}^L_{c^*}\right] &= -i\tilde{X}^L_{c^*} \end{aligned} \tag{4}$$

The quantization form, dual to $\tilde{X}^L_\zeta$ is

$$\begin{aligned} \Theta = \frac{i}{2}[c^*dc &- cdc^*] - cc^*\frac{d\eta}{i\eta} + \frac{d\zeta}{i\zeta} \\ (d\Theta = d\Theta_{PC} &= idc^* \wedge dc - \omega dH \wedge dt) \end{aligned} \tag{5}$$

The characteristic subalgebra is generated by the time generator: $\mathcal{G}_C =< \tilde{X}^L_\eta >$ and a (full and symplectic) polarization is $\mathcal{P} =< \tilde{X}^L_\eta, \tilde{X}^L_c >$. The wave functions are then solutions of the following equations:

$$\begin{aligned} \Xi.\Psi &= i\Psi \Rightarrow \Psi(\zeta,\eta,c,c^*) = \zeta\Phi(\eta,c,c^*) \\ \tilde{X}^L_c.\Psi = 0 &\Rightarrow \Phi = e^{-\frac{cc^*}{2}}\varphi(c^*,\eta) \\ \tilde{X}^L_\eta.\Psi = 0 &\Rightarrow i\frac{\partial\varphi}{\partial t} = \omega c^*\frac{\partial\varphi}{\partial c^*} \end{aligned}$$

with the general expression

$$\Phi = e^{-\frac{cc^*}{2}}\sum_n A_n c^{*n} e^{-in\omega t} \tag{6}$$

In particular, coherent states turn out to be

$$|c> \equiv \sum_{n=0}^{\infty}\Phi_n(c,c^*)^*|n> \;,\; \Phi_n = e^{-\frac{|c|^2}{2}}\frac{c^{*n}}{\sqrt{n!}}\,,\;\; |n> \equiv \frac{(\hat{c}^\dagger)^n|0>}{\sqrt{n!}}\,, \tag{7}$$

where $\hat{c}^\dagger \equiv -\tilde{X}^R_c$. Notice that the vacuum $|0> \equiv e^{-\frac{cc^*}{2}}$ is characterized by being nullified by $\hat{E} \equiv i\omega\tilde{X}^R_\eta$ and $\hat{c} \equiv \tilde{X}^R_{c^*}$.

2. Algebraic Anomalies

In Sec.2, we introduced the concept of full and symplectic polarization subalgebra intended to reduce the Bohr representation. It contains half of the symplectic variables as well as the entire characteristic subalgebra. If the full reduction is achieved, the whole set of physical operators can be rewritten in terms of the basic ones, i.e. those associated with the symplectic variables or, in other words, the operators outside the characteristic subalgebra. For instance, the energy operator for the free particle can be written as $\frac{\hat{p}^2}{2m}$, the angular momentum in 3+1 dimensions is the vector product $\hat{\mathbf{x}} \times \hat{\mathbf{p}}$, or the energy for the harmonic oscillator is $\hat{C}^\dagger\hat{C}$.

However, the existence of a full and symplectic polarization is not guaranteed. We define an **anomalous** group [7] as a a central extension $\tilde{G}$ which do not admit a full and symplectic polarization for certain values of the (pseudo-)cohomology parameters, called the **classical** values of the anomaly. Anomalous groups feature another set of values of the (pseudo-)cohomology parameters, called the **quantum** values of the anomaly, for which the carrier space associated with a full and symplectic polarization contains an invariant subspace. For the classical values of

the anomaly, the classical solution manifold undergoes a reduction in dimension thus increasing the number of (non-linear) relationships among Noether invariants, whereas for the quantum values the number of basic operators decreases on the invariant (reduced) subspace due to the appearance of (higher-order) relations among the quantum operators.

We must remark that the anomalies we are dealing with in this paper are of *algebraic* character in the sense that they appear at the Lie algebra level, and must be distinguished from the *topologic anomalies* which are associated with the non-trivial homotopy of the (reduced) phase space [9].

The non-existence of a full and/or symplectic polarization is traced back to the presence in the characteristic subalgebra associated with certain values of the (pseudo-)cohomology (the classical values of the anomaly) of some element the adjoint action of which is not diagonalizable in the "$x-p$" algebra subspace. The anomaly problem here presented parallels that of the non-existence of invariant polarizations in the Kirillov-Kostant co-adjoint orbits method [10], and the conventional anomaly problem in Quantum Field Theory which manifests itself through the appearance of central charges in the quantum current algebra, absent from the classical (Poisson bracket) algebra [11].

The full reduction of representations in the anomalous case will be achieved by means of a generalized concept (higher-order) polarization (see below).

The Schrödinger group

To illustrate the Lie algebra structure of an anomalous symmetry, let us consider the example of the Schrödinger group. It is the symmetry of the Schrödinger equation for generic potential Ax^2+Bx+C and its Lie algebra (a central extension indeed) has a very simple realization as the Poisson subalgebra [12]

$$\begin{array}{c}\{1,x,p,p^2,x^2,x\cdot p\}\\ \textit{or}\\ \{1,x,p,p^2+x^2,x^2,x\cdot p\}\end{array} \tag{8}$$

The two trivially equivalent versions of the same algebra reflect the existence of two non-equivalent classes of representations, the first one supported on the wave functions of the free particle, the second on the wave functions of the harmonic oscillator.

Let us consider the harmonic oscillator-like version of the Schrödinger algebra and define, accordingly, coordinates c, c^* in the standard way: $x\equiv\frac{1}{\sqrt{2m\omega}}(c+c^*), p\equiv\frac{-im\omega}{\sqrt{2m\omega}}(c-c^*)$, in terms of which the classical Poisson-algebra generators are written:

$$\{1,c,c^*,cc^*,c^2,c^{*2}\}\equiv\{1,c,c^*,\eta,z,z^*\} \tag{9}$$

where we have denoted the quadratic generators by linear, independent variables since all the generators in an abstract algebra are independent. We can think of the

Schrödinger algebra as that of the harmonic oscillator where the $sl(2,R)$ algebra substitutes the $u(1)$ algebra associated with time.

Let us write the Poisson brackets to analyse the anomalous structure:

$$\begin{array}{rcrrcr} \{\eta,c\} &=& -c & \{c^*,c\} &=& 1 \\ \{\eta,c^*\} &=& c^* & \{z^*,z\} &=& -\frac{1}{2}\eta \\ \{\eta,z\} &=& -2z & \{c,z\} &=& 0 \\ \{\eta,z^*\} &=& 2z^* & \{c^*,z^*\} &=& 0 \\ \{c,z^*\} &=& \frac{1}{\sqrt{2}}c^* & \{c^*,z\} &=& \frac{1}{\sqrt{2}}c \end{array} \tag{10}$$

The last line above prevents the existence of a full and symplectic polarization; we can find only a symplectic (non-full) polarization (and the conjugate one) which in terms of the corresponding left-invariant vector fields [7] becomes

$$\mathcal{P} =< \tilde{X}^L_c, \tilde{X}^L_\eta, \tilde{X}^L_z > , \tag{11}$$

and a full (non-symplectic) polarization

$$\mathcal{P}_C =< \tilde{X}^L_\eta, \tilde{X}^L_z, \tilde{X}^L_{z^*} > , \tag{12}$$

Quantizing with the non-full polarization (11) results in a breakdown of the naively expected correspondence between the operators $\hat{z}, \hat{z}^\dagger$ and the basic ones:

$$\begin{array}{ccc} \hat{z} & \not\sim & \hat{c}^2 \\ \hat{z}^\dagger & \not\sim & \hat{c}^{\dagger 2} \end{array} \tag{13}$$

Unlike in the classical case, the operators $(\hat{\eta}), \hat{z}, \hat{z}^\dagger$ behave independently of $\hat{c}, \hat{c}^\dagger$. The operators $\hat{z}, \hat{z}^\dagger$ seem to have symplectic content as if they were canonically-conjugate (basic) operators.

The quantization with the non-symplectic polarization (12) leads to an unconventional representation in which the wave functions depend on both c and c^*, but it is nevertheless irreducible. The operators $\hat{z}, \hat{z}^\dagger$, neither, are expressed in terms of $\hat{c}, \hat{c}^\dagger$.

The Virasoro group

Let us comment very briefly on the relevant, although less intuitive, example of the infinite-dimensional Virasoro group. Its Lie algebra can be written as

$$\left[\tilde{X}^L_{l_n}, \tilde{X}^L_{l_m}\right] = -i(n-m)\tilde{X}^L_{l_{n+m}} - \frac{i}{12}(cn^3 - c'n)\Xi\,, \tag{14}$$

where c parametrizes the central extensions and c' the pseudo-extensions. As is well known, for the particular case in which $\frac{c'}{c} = r^2\,, r \in N, r > 1$, the co-adjoint orbits are not Kählerian. In the present approach, this case shows up as an algebraic anomaly. In fact, the characteristic subalgebra is given by $\mathcal{G}_C =< \tilde{X}^L_{l_0}, \tilde{X}^L_{l_{-r}}, \tilde{X}^L_{l_{+r}} >$, which is not fully contained in the non-full (but symplectic) polarization $\mathcal{P}^{(r)} =<$

$\tilde{X}^L_{l_{n\leq 0}} >$. A detailed description of the representations of the Virasoro group can be found in [6] and references therein.

3. Higher-order Polarizations

In general, to tackle situations like those mentioned above, it is necessary to generalize the notion of polarization. Let us consider the universal enveloping algebra of left-invariant vector fields, $\mathcal{U}\tilde{\mathcal{G}}^L$. We say that a subalgebra $\mathcal{A}$ of $\mathcal{U}\tilde{\mathcal{G}}^L$ is **horizontal** if it does not contain the vertical generator X_0.

Then we define a **higher-order polarization** $\mathcal{P}^{HP}$ as a maximal horizontal subalgebra of $\mathcal{U}\tilde{\mathcal{G}}^L$. With this definition a higher-order polarization contains the maximum number of conditions compatible with the equivariance condition of the wave functions and with the action of the physical operators (right-invariant vector fields).

We notice that now the vector space of functions annihilated by a maximal higher-order polarization is not, in general, a ring of functions and therefore there is no corresponding foliation; that is, they cannot be characterized by saying that they are constant along submanifolds. If this were the case, it would mean that the higher-order polarization was the enveloping algebra of a first-order polarization and, accordingly, we could consider the submanifolds associated with this polarization. In this sense the concept of higher-order polarization generalizes and may replace that of first-order polarization.

We arrive at the formulation of our main general theorem, which was firstly proven for the particular case of the Virasoro group in Ref. [6] and more generally in Ref. [13]

Theorem: *Let $\mathcal{P}^{HO}$ be a higher-order polarization on $\tilde{G}$. On subspaces characterized by*

$$L_{X_0}\psi = i\psi\,, \quad A.\psi = 0 \;\; \forall A \in \mathcal{P}(polarization) \tag{15}$$

all the right-invariant vector fields $\tilde{X}^R$ act irreducibly. Therefore the present quantization procedure gives rise to an irreducible representation of the group $\tilde{G}$, provided it is connected and simply connected.

The definition of higher-order polarization given above is quite general. In all studied examples higher-order polarizations adopt a more definite structure closely related to given first-order (non-full and/or non-symplectic) ones. According to the until now studied cases higher-order polarizations can be given a more operative definition: *A higher-order polarization is a maximal horizontal subalgebra of $\mathcal{U}\tilde{\mathcal{G}}^L$ the vector field content of which is a first order polarization.*

To see how a higher-order polarization operates in practice, we come back to the case of the Schrödinger group and the representation associated with the non-full polarization (11) for which the operators $\hat{z}, \hat{z}^\dagger$ are basic. However, for a particular value (the quantum value of the anomaly) $k = \frac{1}{4}$, the representation of the Schrödinger group *becomes reducible*, although not completely reducible and, *on*

the invariant subspace, the operators $\hat{z}, \hat{z}^\dagger$ do really express as $\hat{c}^2, \hat{c}^{\dagger 2}$ except for numerical proportionality constants. The invariant subspace is constituted by the solutions of a second-order polarization which exists only for $k = \frac{1}{4}$:

$$\mathcal{P}^{HO} =< \tilde{X}^L_c, \tilde{X}^L_\eta, \tilde{X}^L_z, \tilde{X}^L_{z^*} - \alpha(\tilde{X}^L_{c^*})^2 > \tag{16}$$

where the constant α is forced to acquire the value $\alpha = \frac{1}{2\sqrt{2}}$. Physical applications of this particular representation are found in Quantum Optics [14] although no reference to the connection between anomalies and the restriction of k has been made.

In a similar way, in the case of the Virasoro group, for particular values of the parameters c, c' or equivalently $c, h \equiv \frac{c-c'}{24}$ given by the Kac formula [15], the "quantum values" of the anomaly, the representations given by the first order non-full (symplectic) polarizations are reducible since there exist invariant subspaces characterized by certain higher-order polarization equations [6]. Note that there is no one-to-one correspondence between the values of c'/c characterizing the coadjoint orbits of the Virasoro group (the classical values of the anomaly) and the values allowed by the Kac formula (the quantum values of the anomaly), a fact which must be interpreted as a breakdown of the notion of classical limit.

Let us remark, finally, that higher-order polarizations can also be applied to non-anomalous groups to obtain a different, although equivalent, realization of an irreducible representation associated with a first-order full and symplectic polarization. This is the case of the configuration-space realization which cannot be obtained by means of first-order polarizations. Thus, for instance, for the free non-relativistic particle and for the harmonic oscillator, the higher-order polarization $\mathcal{P}^{HO} =< \tilde{X}^L_p, \tilde{X}^L_t - \frac{i\hbar}{2m}(\tilde{X}^L_x)^2 >$ leads to the corresponding quantizations in configuration space [7]. In the relativistic case infinite-order polarizations are required [8,16]. The *modus operandi* in these cases, according to the operative definition of higher-order polarization, is to start with a non-full symplectic polarization containing the generators associated with the variables canonically conjugated to x (or the ones associated with any other realization), $\mathcal{P} =< \tilde{X}^L_p >$ in this case, and complete it with elements of the left enveloping algebra to substitute each one of the generators of the characteristic algebra lacking in the original first-order polarization, in such a way that all of them close a horizontal subalgebra; $\tilde{X}^L_t - \frac{i\hbar}{2m}(\tilde{X}^L_x)^2$ substitutes to $\tilde{X}^L_t$ in this case. In a more general (relativistic) system $\tilde{X}^L_t$ would be substituted by an infinite series in the left enveloping algebra starting by the first-order term $\tilde{X}^L_t$ [16].

Acknowledgment

This work was partially supported by the Dirección General de Investigación Científica y Técnica.

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